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A singular non-local problem at resonance

Nickolai Kosmatov

Department of Mathematics and Statistics, University of Arkansas at Little Rock, Little Rock, AR 72204-1099, USA

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ABSTRACT

We study a second order non-local problem using the coincidence degree theory. We show the existence of a solution whose derivative is singular at the right end-point of the interval, which is a new result for any resonant non-local problem. Under the non-local condition, we find a general way to ensure that the differential operator is a Fredholm operator of index zero.

1. Introduction

In [1], O'Regan and Ma considered the nonresonant multi-point boundary value problem

$$x''(t) = f(t, x(t), x'(t)) + e(t), \quad \text{a.e. } t \in (0, 1),$$

$$x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i),$$

where $m \geq 3$, $0 < \xi_1 < \cdots < \xi_{m-2} < 1$, and f satisfies the Carathéodory conditions and e is a locally Lebesgue-integrable function such that $(1-t)e$ is Lebesgue integrable. The existence of a solution (that has a singular derivative at $t = 1$) was shown by means of the Leray–Schauder continuation principle.

Motivated by [1] we study the non-local problem

$$u''(t) = f(t, u(t), u'(t)), \quad \text{a.e. } t \in (0, 1),$$

$$u'(0) = 0, \qquad \sum_{i=1}^{n} \rho_i u(t_i) = 0,$$

where $n \geq 2$, $0 \leq t_1 < \cdots < t_n \leq 1$, and

$$\sum_{i=1}^{n} \rho_i = 0.$$

We also assume for now that f satisfies the Carathéodory conditions and is bounded (on a bounded subset of $\mathbf{R}^2$) by a function belonging to the same class as e in [1].

With the condition above, the solution space of this problem with $f \equiv 0$ is nontrivial. The coincidence degree theory of Mawhin [2] is used to show the existence of a solution. We note just a few applications of the coincidence degree theory that

E-mail address: nxkosmatov@ualr.edu.

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can be found in [3–6,2]. In addition, [7] is noteworthy as a recent development combining the coincidence degree theory with a cone-theoretic approach (also, see [5,8]). For other results on resonant boundary value problems we refer the reader to [9]. Our results complement and extend the results of [10], which is one of the most general treatments of a second order problem at resonance known to the author.

2. Preliminaries

Definition 2.1. Let X and Z be real normed spaces. A linear mapping $L\colon \operatorname{dom} L \subset X \to Z$ is called a Fredholm mapping if the following conditions hold:

(i) $\ker L$ has a finite dimension, and
(ii) $\operatorname{Im} L$ is closed and has a finite co-dimension.

If L is a Fredholm mapping, its (Fredholm) *index* is the integer $\operatorname{Ind} L = \dim \ker L - \operatorname{codim} \operatorname{Im} L$.

In view Definition 2.1, we will need continuous projectors $P\colon X \to X$ and $Q\colon Z \to Z$ such that

$$\operatorname{Im} P = \ker L, \qquad \ker Q = \operatorname{Im} L, \qquad X = \ker L \oplus \ker P, \qquad Z = \operatorname{Im} L \oplus \operatorname{Im} Q$$

and that the mapping

$$L|_{\operatorname{dom} L \cap \ker P}\colon \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$$

is one-to-one and onto. The inverse of $L|_{\operatorname{dom} L\cap \ker P}$ we denote by $K_P\colon \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$. The generalized inverse of L denoted by $K_{P,Q}\colon Z \to \operatorname{dom} L \cap \ker P$ is defined by $K_{P,Q} = K_P(I - Q)$.

If L is a Fredholm mapping of index zero, then, for every isomorphism $J\colon \operatorname{Im} Q \to \ker L$, the mapping $JQ + K_{P,Q}\colon Z \to \operatorname{dom} L$ is an isomorphism and

$$(JQ + K_{P,Q})^{-1}u = (L + J^{-1}P)u, \quad u \in \operatorname{dom} L.$$

Definition 2.2. Let $L\colon \operatorname{dom} L \subset X \to Z$ be a Fredholm mapping, E be a metric space, and $N\colon E \to Z$ be a mapping. We say that N is L-compact on E if $QN\colon E \to Z$ and $K_{P,Q}N\colon E \to X$ are continuous and compact on E. In addition, we say, that N is L-completely continuous if it is L-compact on every bounded $E \subset X$.

The existence of a solution of the equation $Lu = Nu$ is due to Theorem IV.13 [2]:

Theorem 2.3. *Let $\Omega \subset X$ be open and bounded, L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) *$Lu \neq \lambda Nu$ for every $(u, \lambda) \in ((\operatorname{dom} L \setminus \ker L) \cap \partial\Omega) \times (0, 1)$;*
(ii) *$Nu \notin \operatorname{Im} L$ for every $u \in \ker L \cap \partial\Omega$;*
(iii) *$\deg(QN|_{\ker L \cap \partial\Omega}, \Omega \cap \ker L, 0) \neq 0$, with $Q\colon Z \to Z$ a continuous projector such that $\ker Q = \operatorname{Im} L$.*

Then the equation $Lu = Nu$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

We introduce the space

$$Z = \{g \in L_{\text{loc}}[0, 1)\colon (1 - t)g \in L^1[0, 1]\},$$

where $L_{\text{loc}}[0, 1)$ is the space consisting of functions that are Lebesgue integrable on every interval $[0, a) \subset [0, 1]$. The space Z is endowed the norm

$$\|g\|_Z = \|(1 - t)g\|_{L^1} = \int_0^1 (1 - t)|g(s)|\, ds.$$

We assume that the function f satisfies the Carathéodory conditions with respect to Z, that is, the following hold:

(C_1) for each $(x, y) \in \mathbb{R}^2$, the mapping $t \mapsto f(t, x, y)$ is Lebesgue measurable;
(C_2) for a.e. $t \in [0, 1]$, the mapping $(x, y) \mapsto f(t, x, y)$ is continuous on $\mathbb{R}^2$;
(C_3) for each $r > 0$, there exists a nonnegative $\alpha_r \in Z$ such that, for a.e. $t \in [0, 1]$ and every (x, y) such that $|(x, y)| \le r$, we have $|f(t, x, y)| \le \alpha_r(t)$.

3. Main results

Consider the differential equation

$$u''(t) = f(t, u(t), u'(t)), \quad \text{a.e. } t \in (0, 1), \tag{1}$$

together with the boundary condition

$$u'(0) = 0 \tag{2}$$

and the non-local condition

$$\sum_{i=1}^{n} \rho_i u(t_i) = 0, \tag{3}$$

where $n \geq 2$, $0 \leq t_1 < \cdots < t_n \leq 1$. In addition, we have the critical condition

$$\sum_{i=1}^{n} \rho_i = 0. \tag{4}$$

We choose $\psi \in Z$ with $\psi(t) \neq 0$ on a subset of $(0, 1)$ of positive measure and such that

$$\kappa = \sum_{i=1}^{n} \rho_i \int_0^{t_i} (t_i - s)\psi(s)\, ds \neq 0. \tag{5}$$

(Set, for example, $\psi(t) = \chi_{[t_{n-1}, t_n]}(t)$, the characteristic function of $[t_{n-1}, t_n]$, so that $\kappa = \frac{1}{2}\rho_n(t_n - t_{n-1})^2 \neq 0$.) Let

$$X = \{u \in C[0, 1] \cap C^1[0, 1): \lim_{t\to 1^-} (1 - t)u'(t) \text{ exists}\}$$

with the norm $\|u\| = \max\{\|u\|_0, \|(1 - t)u'\|_0\}$, where $\|\cdot\|_0$ is the max-norm and $\|(1 - t)v\|_0 = \sup_{t\in[0,1)} |(1 - t)v(t)|$. Let $AC_{\text{loc}}[0, 1)$ be the space consisting of functions that are absolutely continuous on every interval $[0, a) \subset [0, 1]$. With

$$\operatorname{dom} L = \{u \in X: u \in AC[0, 1],\ u' \in AC_{\text{loc}}[0, 1),\ u'' \in Z,\ \text{and } u \text{ satisfies (2) and (3)}\},$$

define the mapping $L: \operatorname{dom} L \subset X \to Z$ with

$$Lu(t) = u''(t)$$

and the mapping $N: X \to Z$ by

$$Nu(t) = f(t, u(t), u'(t)).$$

The non-local problem (1)–(3) can now be written as $Lu = Nu$.

We will use a result of [1].

Lemma 3.1. *Let $g \in Z$, then*

$$\int_0^t g(s)\, ds \in L^1[0, 1] \quad \text{and} \quad \lim_{t\to 1^-} (1 - t) \int_0^t g(s)\, ds = 0.$$

In the next result, we obtain suitable decompositions of the spaces X and Z by an exact pair (P, Q) of projectors.

Lemma 3.2. *The mapping $L: \operatorname{dom} L \subset X \to Z$ is a Fredholm mapping of index zero.*

Proof. It is clear that $\ker L = \mathbf{R}$.

Recall (5) and set

$$Qg(t) = \frac{1}{\kappa} \sum_{i=1}^{n} \rho_i \int_0^{t_i} (t_i - s)g(s)\, ds\, \psi(t).$$

Note that $Q: Z \to Z$ is well-defined and is bounded. Indeed,

$$\begin{aligned} |Qg(t)| &\leq \frac{1}{|\kappa|} \sum_{i=1}^{n} |\rho_i| \int_0^{t_i} (t_i - s)|g(s)|\, ds\, |\psi(t)| \leq \frac{1}{|\kappa|} \sum_{i=1}^{n} |\rho_i| \int_0^{1} (1 - s)|g(s)|\, ds\, |\psi(t)| \\ &= \frac{1}{|\kappa|} \sum_{i=1}^{n} |\rho_i| \|g\|_Z\, |\psi(t)| \end{aligned} \tag{6}$$

and

$$\|Qg\|_Z \leq \int_0^1 (1 - t)|Qg(t)|\, dt \leq \frac{1}{|\kappa|} \sum_{i=1}^{n} |\rho_i| \|g\|_Z \int_0^1 (1 - t)|\psi(t)|\, dt = \frac{1}{|\kappa|} \sum_{i=1}^{n} |\rho_i| \|\psi\|_Z \|g\|_Z.$$

In addition, it is straightforward to verify that $Q^2 g = Qg$, $g \in Z$.

Let $g \in \operatorname{Im} L$. Then there exists $u \in \operatorname{dom} L$ with $u''(t) = g(t)$ a.e. $t \in [0, 1)$. By the boundary condition (2),

$$u'(t) = \int_0^t g(s)\, ds, \quad t \in [0, 1),$$

and

$$u(t) = \int_0^t (t-s)g(s)\, ds + c_0, \quad c_0 \in \mathbf{R},$$

for $t \in [0, 1]$. This, by (3) and (4), implies that

$$\sum_{i=1}^{n} \rho_i \int_0^{t_i} (t_i - s)g(s)\, ds = 0. \tag{7}$$

Hence $\operatorname{Im} L \subseteq \{g \in Z\colon (7) \text{ holds}\}$.

Conversely, let $c_0 \in \mathbf{R}$ and

$$u(t) = \int_0^t (t-s)g(s)\, ds + c_0, \quad c_0 \in \mathbf{R},$$

where $g \in Z$ satisfies (7). Then $\lim_{t\to 1^-}(1-t)u'(t) = 0$ by Lemma 3.1. Moreover, (2) holds and $u' \in AC_{\text{loc}}[0, 1)$. Hence $u'' = g$. In view of (4) and (7), we have (3) fulfilled. That is, $u \in \operatorname{dom} L$, which proves that $\{g \in Z\colon (7) \text{ holds}\} \subseteq \operatorname{Im} L$.

We have that

$$\operatorname{Im} L = \{g \in Z\colon (7) \text{ holds}\} = \ker Q.$$

Now, $Z = \operatorname{Im} L + \operatorname{Im} Q$ since $g = (g - Qg) + Qg$, where $g - Qg \in \ker Q$ and $Qg \in \operatorname{Im} Q$ for all $g \in Z$. If $g \in \operatorname{Im} L \cap \operatorname{Im} Q$ and $g \neq 0$. Then $g = c\psi$ for some $c \in \mathbf{R}$, $c \neq 0$. So, $Q(c\psi) = c$, which is a contradiction. Therefore, $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. Moreover, $\operatorname{Im} L$ is closed by a standard dominated convergence argument with account of Lemma 3.1 and $\operatorname{codim} \operatorname{Im} L = \dim \operatorname{Im} Q = \dim \operatorname{span}\{\psi\} = 1$.

Note that $\operatorname{Ind} L = \dim \ker L - \operatorname{codim} \operatorname{Im} L = 0$, that is, L is a Fredholm mapping of index zero.

Define $P\colon X \to X$ by $Pu(t) = u(0)$ with $P^2u = Pu$. Now, $P\colon X \to X$ is a continuous linear projector with $\ker P = \{u \in X\colon u(0) = 0\}$ and

$$\|Pu\| = |u(0)|. \quad \square \tag{8}$$

Since $\operatorname{Im} Q = \operatorname{span}\{\psi\}$ we can, in view of (5), "manipulate" ψ so to satisfy

$$\frac{1}{|\kappa|} \sum_{i=1}^{n} |\rho_i| \|\psi\|_Z = 1. \tag{9}$$

The above, however, is only done to simplify the estimates of Lemma 3.3, which is the only merit of such an assumption.

Define $K_P\colon \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ by

$$K_P g(t) = \int_0^t (t-s)g(s)\, ds, \quad t \in [0, 1].$$

For $g \in \operatorname{Im} L$, $LK_P g(t) = g(t)$. If now $u \in \operatorname{dom} L \cap \ker P$, we have $u(0) = u'(0) = 0$ and

$$K_P Lu(t) = \int_0^t (t-s)u''(s)\, ds = -tu'(0) - u(0) + u(t) = u(t).$$

Therefore,

$$K_P = \left(L|_{\operatorname{dom} L \cap \ker P}\right)^{-1}.$$

Moreover,

$$\|K_P g\|_0 = \max_{t\in[0,1]} |K_P g(t)| \le \max_{t\in[0,1]} \int_0^t (t-s)|g(s)|\, ds \le \|g\|_Z$$

and

$$\|(1-t)(K_P g)'\|_0 = \sup_{t\in[0,1)} \left|(1-t)(K_P g)'(t)\right| \le \sup_{t\in[0,1)} (1-t) \int_0^t |g(s)|\, ds \le \sup_{t\in[0,1)} \int_0^t (1-s)|g(s)|\, ds \le \|g\|_Z.$$

Hence

$$\|K_P g\| \le \|g\|_Z. \tag{10}$$

We have $QN: X \to Z$ written for convenience as $(QN)u(t) = (Q_0 Nu)\psi(t)$, where the functional $Q_0 : Z \to \mathbf{R}$ is defined by

$$Q_0 g = \frac{1}{\kappa}\sum_{i=1}^{n}\rho_i \int_0^{t_i}(t_i - s)g(s)\,ds$$

and $K_{P,Q}N: E \to X$, $E \subset X$, given by

$$K_{P,Q}Nu(t) = K_P(I-Q)Nu(t) = K_P(Nu - (Q_0 Nu)\psi)(t).$$

Note that

$$(K_{P,Q}Nu)'(t) = \int_0^t (Nu(s) - (Q_0 Nu)\psi(s))\,ds.$$

The fact that the above are well-defined follows from the assumptions on f and the mapping properties of Q and K_P. It suffices to recall that $Q_0 : X \to \mathbf{R}$ is well-defined and satisfies

$$|Q_0 Nu| \le \frac{1}{|\kappa|}\sum_{i=1}^{n}|\rho_i|\|Nu\|_Z \le \|\psi\|_Z^{-1}\|\alpha_r\|_Z$$

by (C_3) and (9). Moreover,

$$\begin{aligned}\lim_{t\to 1^-}(1-t)|(K_{P,Q}Nu)'|(t) &\le \lim_{t\to 1^-}\left((1-t)\int_0^t (|Nu(s)| + |Q_0 Nu|\psi(s))\,ds\right)\\ &= \lim_{t\to 1^-}\left((1-t)\int_0^t \left(\alpha_r(s) + \|\psi\|_Z^{-1}\|\alpha_r\|_Z|\psi(s)|\right)ds\right)\\ &= 0\end{aligned}$$

by (C_3) and Lemma 3.1 since

$$\widetilde{\alpha}_r = \alpha_r + \|\psi\|_Z^{-1}\|\alpha_r\|_Z|\psi| \in Z. \tag{11}$$

Lemma 3.3. *The mapping N is L-completely continuous.*

Proof. Let $E \subset X$ be bounded and $\{u_k\} \subset E$. Define the sequence $\{v_k\}$ by $v_k(t) = K_{P,Q}Nu_k(t)$. Set $r = \sup\{\|u\|: u \in E\}$. By (C_3), there exists a function $\alpha_r \in Z$ such that, for all $k \in \mathbf{N}$ and *a.e.* $t \in [0, 1]$,

$$|Nu_k(t)| = |f(t, u_k(t), u_k'(t))| \le \alpha_r(t).$$

For $u \in E$, the mappings QN and $K_P N$ satisfy

$$|QNu_k(t)| \le \|\psi\|_Z^{-1}\|\alpha_r\|_Z|\psi(t)|,$$

and

$$|K_P Nu_k(t)| \le \|\alpha_r\|_Z,$$

which are obtained the same way as (6) and (10) using (9). Using (11), we combine the above as

$$|Nu_k(t) - QNu_k(t)| \le \widetilde{\alpha}_r(t).$$

For $t \in [0, 1]$ and $k \in \mathbf{N}$,

$$|v_k(t)| = |K_{P,Q}Nu_k(t)| = |K_P(Nu_k - QNu_k)(t)| \le K_P\widetilde{\alpha}_r(t) \le \|\widetilde{\alpha}_r\|_Z = 2\|\alpha_r\|_Z,$$

that is, the sequence $\{v_k\}$ is bounded in $C[0, 1]$. Also, for $t \in [0, 1)$, setting $w_k(t) = (1-t)v_k'(t)$,

$$|w_k(t)| = |(1-t)(K_{P,Q}Nu_k)'(t)| = \left|(1-t)\int_0^t (Nu_k(s) - QNu_k(s))\,ds\right| \le \int_0^1 (1-s)\widetilde{\alpha}_r(s)\,ds \le \|\widetilde{\alpha}_r\|_Z,$$

that is, the sequence $\{(1-t)v_k'\}$ is bounded in $C[0, 1)$ and

$$\lim_{t\to 1^-}(1-t)v_k'(t) = 0.$$

So, $\{v_k\}$ is bounded in X.

Let $t_1, t_2 \in [0, 1]$ and $t_1 < t_2$. Then

$$\begin{aligned}|v_k(t_2) - v_k(t_1)| = |K_{P,Q}Nu_k(t_2) - K_{P,Q}Nu_k(t_1)| &= \left|\int_{t_1}^{t_2} (K_{P,Q}Nu_k)'(s)\, ds\right| \\ &\le \left|\int_{t_1}^{t_2}\int_0^s (Nu_k(\tau) - QNu_k(\tau))\, d\tau\, ds\right| \\ &\le \int_{t_1}^{t_2}\int_0^s \widetilde{\alpha}_r(s)\, d\tau\, ds.\end{aligned}$$

Since, by Lemma 3.1,

$$\int_0^{\cdot} \widetilde{\alpha}_r(\tau)\, d\tau \in L^1[0, 1],$$

$\{v_k\}$ is equicontinuous by the absolute continuity of antiderivative.

Next $w_k'(t) = -v_k'(t) + (1-t)v_k''(t)$ and

$$\begin{aligned}|w_k'(t)| &\le |v_k'(t)| + (1-t)|v_k''(t)| = \left|\int_0^t (Nu_k(s) - QNu_k(s))\, dst\right| + (1-t)|Nu_k(t)| \\ &\le \int_0^t \widetilde{\alpha}_r(s)\, ds + (1-t)\alpha_r(t),\end{aligned}$$

where the function above is denoted by $\mu(t)$. By Lemma 3.1, $\mu \in L^1[0, 1]$. Hence

$$|w_k(t_2) - w_k(t_1)| \le \left|\int_0^{t_2} w_k'(s)\, ds - \int_0^{t_1} w_k'(s)\, ds\right| \le \int_{t_1}^{t_2} |w_k'(s)|\, ds \le \int_{t_1}^{t_2} \mu(s)\, ds$$

which shows that $\{w_k\}$ is equicontinuous in $[0, 1)$.

Since $\{v_k\} \in C[0, 1]$ is bounded and equicontinuous, we can assume without loss of generality that it uniformly converges to some $v_0 \in C[0, 1]$ (otherwise, we pass to a convergent subsequence by the Arzela–Ascoli Theorem). Similarly, $\{w_k\}$ converges uniformly to some $w_0 \in C[0, 1)$ with $\lim_{t\to 1^-}(1-t)w_0(t) = 0$. Moreover, $w_0(t) = (1-t)v_0'(t)$, $t \in [0, 1)$. Consequently, $K_{P,Q}N(E)$ is relatively compact. Since the function $f: [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ satisfies the Carathéodory conditions with respect to Z, $K_{P,Q}N: E \to X$ is continuous by the Lebesgue dominated convergence theorem.

By similar arguments, $QN: E \to Z$ is continuous. Furthermore, $QN(E) \subset Z$ is relatively compact by Riesz's compactness criterion. Now, since the mappings QN and $K_{P,Q}N$ are compact on an arbitrary bounded $E \subset X$, the mapping $N: X \to Z$ is L-completely continuous by Definition 2.2. $\square$

We will show the existence of a solution under the following assumptions:

(H_1) there exists a constant $A > 0$ such that $u \in \operatorname{dom} L \setminus \ker L$ with $|u(t)| > A$ implies $QNu(t) \neq 0$ on $(0, 1]$;

(H_2) there exist $\beta, \chi, \delta \in Z$, $\gamma \in L^1[0, 1]$ and a continuous nondecreasing function $\phi: [0, \infty) \to [0, \infty)$ and $x_0 > 0$ with the properties:

(a)
$$\|\beta\|_Z + \|(1-t)^{-1}\gamma\|_Z < \frac{1}{2};$$

(b) for all $x \ge x_0$
$$x \ge \frac{A + 2\|\delta\|_Z}{1 - 2(\|\beta\|_Z\| + \|(1-t)^{-1}\gamma\ \|_Z)} + \frac{2\|\chi\|_Z}{1 - 2(\|\beta\|_Z + \|(1-t)^{-1}\gamma\|_Z)}\phi(x); \tag{12}$$

(c) $f: [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ satisfies
$$|f(t, x, y)| \le \delta(t) + \beta(t)|x| + \gamma(t)|y| + \chi(t)\phi(|x|);$$

(H_3) there exists a constant $B > 0$ such that, for every $c \in \mathbf{R}$ satisfying $|c| > B$, we have $\operatorname{sgn}\,[cQ_0Nu_c] \neq 0$, where $u_c(t) = c$.

Theorem 3.4. *If the hypotheses* (H_1)–(H_3) *are satisfied, then the non-local problem* (1)–(4) *with* (5) *has a solution.*

Proof. Let $\Omega_1 = \{u \in \operatorname{dom} L \setminus \ker L: Lu = \lambda Nu$ for some $\lambda \in (0, 1)\}$. Applying (H_1), $QNu(t) = 0$ for all $t \in [0, 1]$. Hence there exists $t_0 \in (0, 1]$ such that $|u(t_0)| \le A$. Then

$$\int_0^{t_0} (t_0 - s)u''(s)\, ds = -t_0u'(0) + u(t_0) - u(0) = u(t_0) - u(0)$$

since $u \in \operatorname{dom} L$. Hence, by (8),

$$\|Pu\| = |u(0)| \le |u(t_0)| + \int_0^{t_0} (t_0 - s)|u''(s)|\, ds \le A + \|Lu\|_Z < A + \|Nu\|_Z.$$

Since $(I-P)u \in \operatorname{dom} L \cap \ker P = \operatorname{Im} K_P$, we have, for $u \in \Omega_1$,

$$\|(I-P)u\| = \|K_pL(I-P)u\| \le \|L(I-P)u\|_Z = \|Lu\|_Z < \|Nu\|_Z$$

by (10). Also, $Pu \in \operatorname{Im} P = \ker L$ and, therefore,

$$\|u\| \le \|Pu\| + \|(I-P)u\| < A + 2\|Nu\|_Z.$$

From (H_2) and the previous inequality, it follows that

$$\begin{aligned}\|u\| &< A + 2\|\delta + \beta|u| + \gamma|u'| + \chi\phi(|u|)\,\|_Z \\ &< A + 2(\|\delta\|_Z + \|\beta\|_Z\|u\|_0 + \|(1-t)^{-1}\gamma\|_Z\|(1-t)u'\|_0 + \|\chi\|_Z\phi(\|u\|_0)) \\ &\le A + 2\|\delta\|_Z + 2(\|\beta\|_Z + \|(1-t)^{-1}\gamma\|_Z)\|u\| + 2\|\chi\|_Z\phi(\|u\|)\end{aligned}$$

and

$$\|u\| < \frac{A + 2\|\delta\|_Z}{1 - 2(\|\beta\|_Z\| + \|(1-t)^{-1}\gamma\,\|_Z)} + \frac{2\|\chi\|_Z}{1 - 2(\|\beta\|_Z + \|(1-t)^{-1}\gamma\|_Z)}\phi(\|u\|).$$

Suppose that Ω_1 is unbounded. Then, in view of (12), we arrive at a contradiction. Therefore, Ω_1 is bounded.

Set $\Omega_2 = \{u \in \ker L\colon Nu \in \operatorname{Im} L\}$. Hence $u_c \in \ker L$ is given by $u_c(t) = c, c \in \mathbf{R}$. Then $(QNu_c)(t) = 0$, since $Nu \in \operatorname{Im} L = \ker Q$. It follows from (H_3) that $\|u_c\| = \max\{\|u_c\|_0, \|(1-t)u_c'\|_0\} = |c| \le B$, that is, Ω_2 is bounded.

Define the isomorphism $J\colon \operatorname{Im} Q \to \ker L$ by $J^{-1}u_c = c\psi$, $u_c(t) = c$, $c \in \mathbf{R}$. Let $\Omega_3 = \{u \in \ker L\colon -\lambda u + (1-\lambda)JQNu = 0,\ \lambda \in [0,1]\}$ if $\operatorname{sgn}[cQ_0Nu_c] = -1$. Then $u \in \Omega_3$ implies that $\lambda c\psi(t) = (1-\lambda)(Q_0Nu_c)\psi(t)$. If $\lambda = 1$, then $c = 0$ and, if $\lambda \in [0,1)$ and $|c| > B$, then $0 < \lambda c^2 = (1-\lambda)cQ_0Nu_c < 0$, which is a contradiction. Let $\Omega_3 = \{u \in \ker L\colon \lambda u + (1-\lambda)Q_0Nu = 0,\ \lambda \in [0,1]\}$ if $\operatorname{sgn}[cQ_0Nu_c] = 1$, and we arrive at a contradiction, again. Thus, $\|u_c\| \le B$, for all $u_c \in \Omega_3$.

Let Ω be open and bounded such that $\cup_{i=1}^3 \overline{\Omega}_i \subset \Omega$. Then the assumptions (i) and (ii) of Theorem 2.3 are fulfilled. Lemma 3.3 shows that the mapping N is L-compact on $\overline{\Omega}$. Lemma 3.2 establishes that L is a Fredholm mapping of index zero.

Define

$$H(u,\lambda) = \pm\lambda u + (1-\lambda)JQNu.$$

By the degree property of invariance under a homotopy, if $u \in \ker L \cap \partial\Omega$, then

$$\begin{aligned}\deg(JQN|_{\ker L\cap\partial\Omega}, \Omega \cap \ker L, 0) &= \deg(H(\cdot,0), \Omega\cap\ker L, 0) \\ &= \deg(H(\cdot,1), \Omega\cap\ker L, 0) = \deg(I, \Omega\cap\ker L, 0) \ne 0.\end{aligned}$$

Finally, the assumption (iii) of Theorem 2.3 is fulfilled and the proof is completed. □

We remark that the condition (c) in (H_2) can be replaced with

$$|f(t,x,y)| \le \delta(t) + \beta(t)|x| + \gamma(t)|y| + \chi(t)\phi(|y|),$$

where $\phi(x) = |x|^\theta, \theta \in (0,1)$. For example, consider (1)–(5) with $\xi_1 = 1/4, \xi_2 = 3/4, \rho_1 = -1, \rho_2 = 1$. Then $\{u : u(t) = c\} \not\subset \operatorname{Im} L$ and we can take $\psi \equiv 1$. If

$$f(t,x_1,y_1) = 1 + \frac{1}{4}x + \frac{1}{8(1-t)^{3/2}}y + \sqrt{|x|},$$

then, for $u_c \equiv c$,

$$cQ_0Nu_c = \frac{1}{\kappa}\sum_{i=1}^n \int_0^{t_i} \rho_i(t_i - s)Nu_c(s)\,ds = c\left(1 + \frac{1}{4}c\right).$$

That is, we can choose $B = 4$. Clearly, $\|\beta\|_Z + \|(1-t)^{-1}\gamma\|_Z < 1/2$.

References

[1] R. Ma, D. O'Regan, Solvability of singular second order m-point boundary value problems, J. Math. Anal. Appl. 301 (2005) 124–134.

[2] J. Mawhin, Topological Degree Methods in Nonlinear Boundary Value Problems, in: NSF–CBMS Regional Conference Series in Math., No. 40, Amer. Math. Soc., Providence, RI, 1979.

[3] W. Feng, J.R.L. Webb, Solvability of three point boundary value problem at resonance, Nonlinear Anal. 30 (1997) 3227–3238.

[4] C.P. Gupta, A second order m-point boundary value problem at resonance, Nonlinear Anal. 24 (1995) 1483–1489.

[5] G. Infante, M. Zima, Positive solutions of multi-point boundary value problems at resonance, Nonlinear Anal. 69 (2008) 2458–2465.

[6] E.R. Kaufmann, A third order non-local boundary value problem at resonance, Electron. J. Qual. Theory Differ. Equ. 1 (2009) 1–11. (Spec. Ed.).

[7] D. O'Regan, M. Zima, Leggett–Williams norm-type theorems for coincidences, Arch. Math. (Basel) 87 (2006) 233–244.

[8] J.R.L. Webb, M. Zima, Multiple positive solutions of resonant and non-resonant non-local boundary value problems, Nonlinear Anal. 71 (2009) 1369–1378.

[9] M. García-Huidobro, C.P. Gupta, R. Manásevich, Solvability for a nonlinear three-point boundary value problem with p-Laplacian-like operator at resonance, Abstr. Appl. Anal. 6 (4) (2001) 191–213.

[10] Z. Zhao, J. Liang, Existence of solutions to functional boundary value problem of second-order nonlinear differential equation, J. Math. Anal. Appl. 373 (2011) 614–634.